UDC 532.529.6

THE LAGRANGE FUNCTION FOR VORTEX FLOWS AND DYNAMICS OF DEFORMED DROPS

PMM Vol. 41, № 1, 1977, pp. 79-94

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(Received June 24, 1976)

The flow of a perfect incompressible fluid in a region bounded by a surface whose position is determined by a finite number of parameters is considered. The Lagrange function dependent on a finite number of variables is determined for the following cases:

1). Potential cyclic motion of fluid with deformable boundary.

2). Cyclic plane-parallel motion of fluid with constant vortex and similar axisymmetric vortex flow.

It is shown that at high Reynolds numbers a velocity field whose vortex is proportional to the distance from the axis of symmetry is established inside a drop in an axisymmetric flow. In the case of vortex flow inside the drop a variational equation for the fixed shape of the drop and the condition of its stability are obtained from the Lagrange equation, and investigated on a set of ellipsoids.

For given vortex intensity and Weber number there generally exist two fixed shapes of the drop. In the case of a liquid drop in gas one of these shapes is always unstable. It is shown that the dynamic head inside the drop can considerably exceed the dynamic head of the external flow and that the complex flow separation around the drop has virtually no effect on the shape of the drop and its stability. Such is the case of free fall of a water drop in air, when it is of shape elongated in a direction parallel to the flow.

1. Certain results of the exact statement of the problem of steady motion of a drop. The steady axisymmetric motion of a drop of incompressible viscous fluid within another viscous incompressible fluid is considered. Let $\mathbf{v}_+'$ and ζ_+' be, respectively, the velocity and vortex vectors of the motion of fluid outside the drop, μ_+ and ν_+ be the dynamic and kinematic viscosities, and ρ_+ the density of fluid outside the drop. The corresponding parameters of motion in the drop are $\mathbf{v}_-'$, ζ_-', μ_-, ν_- and ρ_-. Cylindrical coordinates x, y, α are used with coordinate x taken along the axis of symmetry, y representing the distance from that axis, and α the angle of elevation. In such system of coordinates only the two first components of velocity vectors are nonzero, and of the vortex vectors ζ_+ and ζ_- only the third components are nonzero. At the drop surface S the conditions of equality of velocities and tangential stresses and the condition of equality of normal stresses and surface tensions at the interface of the two media

$$\mathbf{v}_+' = \mathbf{v}_-', \quad v_+' = |\mathbf{v}_+'|, \quad v_-' = |\mathbf{v}_-'| \tag{1.1}$$

$$\mu_+ (\zeta_+' + 2\varkappa v'_+) = \mu_- (\zeta_-' + 2\varkappa v_-')$$

$$\overset{+}{p}_{nn} - \overset{-}{p}_{nn} = 2H\sigma$$

where $\varkappa$ is the curvature of the generatrix of surface S, H is the mean curvature of surface S, and σ is the surface tension coefficient — are satisfied.

In the system of coordinates attached to the drop, velocity $\mathbf{v}_+'$ at infinity must tend to a constant vector $\mathbf{u}$ (velocity of fluid at infinity).

The Reynolds number R_+ of the external flow is assumed high, hence outside the thin boundary layer $\delta \sim l \,/\, \sqrt{R_+}$ (l is a characteristic dimension of the drop) velocity $\mathbf{v}_+'$ is close to velocity $\mathbf{v}_+$ in a potential flow.

A velocity field whose vortex vector is everywhere nonzero sets inside the drop. It can be shown that at the limit $\nu_- \to 0$ a velocity field whose elevation component of the vortex vector is proportional to the distance from the axis of symmetry

$$\zeta_-' \to \zeta = y\omega, \quad \omega = \text{const} \tag{1.2}$$

obtains inside the drop. This is so, since in cylindrical coordinates $\omega' = \zeta' \,/\, y$ satisfies the equation

$$(\mathbf{v}_-'\nabla)\,\omega' = \nu_-\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{3}{y}\frac{\partial}{\partial y}\right)\omega' \tag{1.3}$$

When $\nu_- \to 0$ the quantity ω' remains constant along the streamline. Since the drop boundary is a streamline, ω' at it tends to become constant. Since in the solutions of Eq. (1.3) the maximum and minimum are reached at the region boundary [1], hence everywhere inside the drop ω' tends to become constant when $\nu_- \to 0$.

Note. 1. 1. The plane case was considered in [2].

Furthermore the Hadamard-Rybczynski solution [3] for slow motion of the drop shows that at high viscosity the quantity ω' is nearly constant.

It is, thus, possible to assume in estimates that inside the drop the motion has a constant vorticity, which implies that at the drop boundary for the velocity and the vortex $\zeta_-' \sim v_-' \,/\, l \sim \omega l$.

The motion of drops can be of two kinds. In the first case $v_-' \ll v_+$ outside the drop close to the boundary S the flow is close to that in the boundary layer δ at the solid boundary. The velocity v_+ across the boundary layer thickness δ varies considerably from v_+ to $v_+' = v_-'$. Outside at the drop boundary the vortex is $\zeta_+ \sim v_+ \,/\, \delta$, hence for the determination of the drop motion we obtain from the boundary condition the following estimate for the characteristics:

$$v_-' \,/\, v_+ \sim \mu_+ \sqrt{R_+} \,/\, \mu_- \ll 1 \tag{1.4}$$

Estimate (1.4) is usually satisfied in the case of liquid drops moving in a gas. Thus for a water drop in the air the ratio $\mu_+ \,/\, \mu_- \approx 0.02$ and for Reynolds numbers $R_+ \sim 100$ it is possible to assume that condition (1.4) is satisfied.

For the fluid kinetic energy outside the drop T_+ and inside it T_- we can obtain from (1.4) the following estimates:

$$\frac{T_-}{T_+} \sim \frac{\rho_-}{\rho_+}\left(\frac{v_-'}{v_+}\right)^2 \sim \frac{\rho_+\nu_+^2}{\rho_-\nu_-^2} R_+ \tag{1.5}$$

Estimates (1.5) imply that in many practically important cases the kinetic energy of the internal motion in the drop is comparable to and may even exceed the kinetic energy of external motion. Thus for a water drop in air $T_- \,/\, T_+ \sim 0.3 \; R_+$.

In the second case

$$\mu_+ \sqrt{R_+} \,/\, \mu_- \gtrsim 1, \; v_-' \sim v_+ \tag{1.6}$$

the velocity of the fluid potential motion outside the drop is of the same order as that inside the drop. In this case the ratio of kinetic energies is of the same order as that of the ratio of densities

$$T_- / T_+ \sim \rho_- / \rho_+ \tag{1.7}$$

Estimates (1.5) and (1.7) show that in both cases of motion of drops in gas the internal motion and the surface tension have a greater effect on the shape and stability of the drop than the external flow. The external flow has, however, a considerable effect on the drag and, consequently, on the free fall velocity. This can be proved with the use of similar estimates for the energy dissipation. Below the velocity of drop motion is assumed known, which simplifies the problem by avoiding the calculation of the complex separation flow.

2. The dynamic model of a drop. The distribution of normal stresses at the drop boundary differs at high Reynolds numbers only slightly from pressures of corresponding flows of perfect fluids. Hence it is possible to investigate the shape and stability of the drop using the theory of perfect fluids. In this approach to the solution of the considered problem it is usual to assume that the flow inside a steadily moving drop as well as outside it, is potential [4 — 6]. This means that motion is absent inside a steadily moving drop. However estimates (1.5) and (1.7) show that the steady internal motion in the drop is the determining factor for the shape and stability of the drop.

At the limit of high Reynolds number a flow of perfect fluid is established inside the drop, where $\mathbf{v}_-$, ζ_- $(0, 0, \zeta_-)$, p_- and ρ_- are, respectively, the velocity and vortex vectors, pressure, and density of the liquid drop. As previously shown by (1.2), the elevation component of the vortex is proportional to the distance from the axis of symmetry.

If separation and the vortex trail are neglected in the flow around the drop, then it is possible to assume, as in [4 — 6], that outside the drop a separation-free potential flow of a perfect incompressible fluid ($\mathbf{v}_+$, p_+, ρ_+ are, respectively, the velocity, pressure and density of fluid outside the drop) takes place around the drop. This assumption is not an important restriction of the flow of gas past the drop, as shown above. At the drop boundary in the case of a viscous fluid conditions (1.1) must be replaced by condition

$$v_{+n} = v_{-n}, \quad p_- - p_+ = 2H\sigma \tag{2.1}$$

which defines the conditions of equality of normal velocities and the jump of pressures.

A solution which satisfies condition (1.2) inside the drop, the condition of flow potentiality outside it, and boundary conditions (2.1) can be obtained for any ω. The vorticity cannot then be determined by means of the theory of perfect fluids, it can only be derived in the case of a viscous fluid by using, for instance, estimates (1.5) and (1.7).

Let the drop surface S be defined by the equation $F(x, y, q_i) = 0$. The generalized coordinates q_i and velocities $q_i^{\cdot}$ $(i = 1, 2, \ldots, N)$ define the position of surface S and the velocity of its motion along the normal. The kinetic energies of fluid inside and outside the drop is then a single-valued function of generalized coordinates q_i and velocities $q_i^{\cdot}$, and also of the quantity that specifies the vorticity inside the drop. Hence it is possible to expect that the Lagrange equations are valid for the motion of the drop surface S.

When the motion inside and outside the drop is potential, these equations can be derived from Euler's equations [7 — 9]. The Lagrange function is the remainder of the kinetic energy of the combined motions inside and outside the drop and of the potential energy. The form of the Lagrange function for vortex motion is not known.

3. The Lagrange function. In conformity with [10] we call the remainders

$$\Delta Q = Q(\mathbf{x} + \xi(\mathbf{x}, t), t) - Q_0(\mathbf{x}, t) \quad \delta Q = Q(\mathbf{x}, t) - Q_0(\mathbf{x}, t) \tag{3.1}$$

respectively, the Lagrange and Euler variations of the characteristic Q that are induced by perturbations. In formulas (3.1) $Q_0(\mathbf{x}, t)$ is the characteristic of the original flow, $Q(\mathbf{x} + \xi, t)$ is the characteristic of the perturbed flow, and $\xi(\mathbf{x}, t)$ is the "infinitely small" Lagrangian translation induced by perturbation. Vectors $\mathbf{x}$ and $\mathbf{x} + \xi(\mathbf{x}, t)$ determine coordinates of one and the same particle of fluid in the original and the perturbed flows at the same instant of time t.

The relation between operations Δ and δ

$$\Delta = \delta + \xi_j \frac{\partial}{\partial x_j} \tag{3.2}$$

is obvious.

Here and in what follows the same subscripts $j = 1,2,3$ indicate summation.

By virtue of incompressibility we have

$$\operatorname{div} \mathbf{v} = 0, \quad \operatorname{div} \xi = 0 \tag{3.3}$$

It follows from the definition of Lagrangian translation ξ that

$$\frac{d\xi}{dt} = \Delta \mathbf{v} = \delta \mathbf{v} + \xi_j \frac{\partial \mathbf{v}}{\partial x_j}, \quad \frac{\partial \xi}{\partial t} + \operatorname{rot}(\xi \times \mathbf{v}) = \delta \mathbf{v} \tag{3.4}$$

Let ψ be the streamfunction of the axisymmetric motion and g a similar function for vector ξ, then

$$\mathbf{v} = \operatorname{rot} \mathbf{k} \frac{\psi}{y}, \quad \xi = \operatorname{rot} \mathbf{k} \frac{g}{y} \tag{3.5}$$

where $\mathbf{k}$ is the unit vector normal to the meridian plane, and functions ψ and g exist in virtue of the incompressibility conditions (3.3).

The velocity field inside the drop satisfies Euler's equation

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p \tag{3.6}$$

With the use of (3.4) we obtain the identity

$$\Delta \frac{1}{2} \rho v^2 = \rho \left(\frac{d}{dt} (\mathbf{v}\xi) - \xi \frac{d\mathbf{v}}{dt} \right)$$

whose integration over the drop volume V and time within the limits from t_1 to t_2 yields

$$\int_{t_1}^{t_2} dt \delta T = \int_V \rho \mathbf{v} \xi \, d\tau \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} dt \int_V \xi \rho \frac{d\mathbf{v}}{dt} d\tau$$

$$T = \int_V \rho \frac{v^2}{2} d\tau, \quad \delta T = \int_V \rho \mathbf{v} \Delta \mathbf{v} \, d\tau$$

Using the equation of motion (3.6) and the condition of incompressibility (3.3) we can transform the integral of $\xi \rho \, d\mathbf{v} / dt$ to a surface one, from which

$$\int_{t_1}^{t_2} dt \delta T = \int_V \rho \mathbf{v} \xi \, d\tau \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} dt \int_S p \delta n \, dS \tag{3.7}$$

$$\delta n = \xi \mathbf{n} = \sum_{i=1}^{N} w_i \delta q_i$$

where δn is the displacement of the drop surface S along the normal when its generalized coordinates are varied by the quantity δq_i. The position functions w_i at the surface define the displacement of surface S along the normal for the unit variation of the i-th generalized coordinate.

Let $q_i(t)$ and $\delta q_i(t)$ be arbitrary functions specified along segment (t_1, t_2). At the ends t_1 and t_2 of the segment the variations of δq_i and, consequently, also the normal displacement δn are zero

$$\delta q_i|_{t_1} = \delta q_i|_{t_2} = 0, \quad \delta n|_{t_1} = \delta n|_{t_2} = 0 \tag{3.8}$$

Equality (3.8) implies that at the initial and final instants of time t_1 and t_2 the surfaces in the original and the perturbed flows are the same.

The condition $\xi|_{t_1} = 0$ may be imposed on the Lagrangian displacement, which implies that in the original and the perturbed flows the coordinates of fluid particles are the same at the initial instant of time t_1. At instant of time t_2 the Lagrangian displacement ξ even for a potential flow is, generally speaking, nonzero, because the position of a fluid particle depends on the whole previous history of the motion of surface S, i.e. the fluid represents a nonholonomic system.

Note 3.1. It is usual to substantiate Lagrange equations by the proof given in the monograph [11]. However that proof is untenable for the case of acyclic potential motion of fluid. The monograph does not define operator D/Dt; if it is meant to be the operator of the Lagrangian characteristic variation Δ, and the virtual velocity V is taken as the Lagrangian displacement ξ (only then is the formula (16) valid in this case and operator D/Dt may be brought into the integrand of the integral over the volume which moves together with the fluid), then formula (15) assumes the form

$$\xi = \sum_i \frac{\partial \mathbf{v}}{\partial q_i^{\cdot}} \delta q_i$$

This formula, which lies at the basis of the whole proof, is valid when the fluid represents a holonomic system, while in the general case of acyclic potential motion it is invalid.

Lagrange equations were obtained in [9] from the variational equation (3.7) in the case of potential motion in a simply connected region. In that case the first integral in the right-hand side of (3.7) reduces to a surface integral which by virtue of (3.8) is zero. For a vortex motion that integral gives a substantial contribution to the Lagrange function. Further deduction is associated with the transformation of the first integral in (3.7), for which it is necessary to introduce the following definition.

Definition. Let the position of surface S be uniquely determined by the generalized coordinates q_i $(i = 1, 2, \ldots, N)$. We shall consider a medium as attached to surface S if the coordinates $\mathbf{x}$ of that medium depend only on the position of surface S.

Vectors $\mathbf{x}$ and $\mathbf{v}^*$ of coordinates and velocity of medium particles are functions of generalized coordinates q_i and velocities $q_i^{\cdot}$, and also of the Lagrangian coordinates $\mathbf{x}^*$ of the medium

$$\mathbf{x} = \mathbf{X}(q_i, \mathbf{x}^*), \quad \mathbf{v}^* = \sum_{i=1}^{N} \frac{\partial \mathbf{X}}{\partial q_i} q_i^{\cdot} \tag{3.9}$$

The Lagrangian displacement ξ^* of the medium may be defined similarly to vector ξ of the moving fluid. For an incompressible medium attached to surface S vectors $\mathbf{v}^*$ and ξ^* satisfy the equations

$$\operatorname{div} \mathbf{v}^* = 0, \quad \operatorname{div} \xi^* = 0, \quad \frac{\partial \xi^*}{\partial t} + \operatorname{rot}(\xi^* \times \mathbf{v}^*) = \delta \mathbf{v}^* \tag{3.10}$$

which are similar to (3.3) and (3.4).

For an axisymmetric motion of an incompressible medium attached to the body we can introduce streamfunctions ψ^* and g^*

$$\mathbf{v}^* = \operatorname{rot} \mathbf{k} \frac{\psi^*}{y}, \quad \xi^* = \operatorname{rot} \mathbf{k} \frac{g^*}{y} \tag{3.11}$$

which are similar to (3.5). The substitution of (3.11) into (3.10) and (3.5) into (3.4) yields the following integrals for Eqs. (3.10) and (3.4):

$$\frac{dg}{dt} = \delta\psi, \quad \frac{d^* g^*}{dt} = \delta\psi^*, \quad \frac{d^*}{dt} = \frac{\partial}{\partial t} + v^*_j \frac{\partial}{\partial x_j} \tag{3.12}$$

$$\frac{\partial g}{\partial t} = \Delta\psi, \quad \frac{\partial g^*}{\partial t} = \Delta^*\psi, \quad \Delta^* = \delta + \xi_j^* \frac{\partial}{\partial x_j}$$

To transform the integral in the variational equation (3.7) we use the identity

$$\mathbf{v}(\xi - \xi^*) = \operatorname{div}\left(\frac{g - g^*}{y} \mathbf{k} \times \mathbf{v}\right) + \frac{g - g^*}{y} \mathbf{k} \operatorname{rot} \mathbf{v}$$

which is easily obtainable from equalities (3.5) and (3.11). From this

$$\int_V \mathbf{v}(\xi - \xi^*)\, d\tau = \int_S \frac{g - g^*}{y} \mathbf{k} \times \mathbf{v}\mathbf{n}\, dS + \int_V \frac{g - g^*}{y} \mathbf{k} \operatorname{rot} \mathbf{v}\, d\tau \tag{3.13}$$

At the boundary S vectors ξ and ξ^*, as well as v and $\mathbf{v}^*$ have the same normal components

$$(\xi - \xi^*)\,\mathbf{n}\,|_S = 0, \quad (\mathbf{v} - \mathbf{v}^*)\,\mathbf{n}\,|_S = 0 \tag{3.14}$$

This implies that at the boundary S the remainders of "stream"-functions $g - g^*$ and $\psi - \psi^*$ are constant. Taking the constant $g - g^*$ out of the surface integral sign, passing to the volume integral, and taking into account that

$$\operatorname{div}\left(\frac{\mathbf{k}}{y} \times \mathbf{v}\right) = -\,\omega, \quad \frac{\mathbf{k}}{y} \operatorname{rot} \mathbf{v} = \omega$$

from (3.13) we obtain

$$\int_V \mathbf{v}(\xi - \xi^*)\, d\tau = -\,\omega V (g - g^*)\,|_S + \int_V (g - g^*)\, \omega\, d\tau$$

Using (3.12) the derivative of the remainder $g - g^*$ with respect to time along the trajectory of particles can be reduced to the form

$$\frac{d(g - g^*)}{dt} = \Delta\psi - \Delta^*\psi^* + v_j \frac{\partial}{\partial x_j}(g - g^*) = \Delta(\psi - \psi^*) + \operatorname{div}(g - g^*)(\mathbf{v} - \mathbf{v}^*)$$

From this with allowance for the second of conditions (3.14)

$$\frac{d}{dt}\int_V \rho\mathbf{v}(\xi - \xi^*)\, d\tau = -\,\rho\omega\Delta(\psi - \psi^*)\,|_S + \delta \int_V \rho\omega(\psi - \psi^*)\, d\tau \tag{3.15}$$

The remainder $\psi - \psi^*$ is determined to within an arbitrary function of time which can be selected so that at the attached boundary $\psi - \psi^*|_S = 0$. Integrating equality (3.15) with respect to time from t_1 to t_2 and taking into consideration that for the medium attached to surface S, $\xi^* = 0$ at instants of time t_1 and t_2 (definition (3.9) implies that the medium attached to the body is holonomic), we obtain

$$\int_V \rho\mathbf{v}\xi\, d\tau\,\Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} dt\rho\omega\delta \int_V (\psi - \psi^*)\, d\tau \tag{3.16}$$

Substituting (3.16) into (3.7) for the motion inside the drop we obtain

$$\int_{t_1}^{t_2} dt\delta\left(T_- - \rho\omega\int_V (\psi-\psi^*)\,d\tau\right) = \int_{t_1}^{t_2} dt\int_S p_-\delta n\,dS \tag{3.17}$$

For the motion outside the drop we have the equality [12]

$$\int_{t_1}^{t_2} dt\delta T_+ = -\int_{t_1}^{t_2} dt\int_S p_+\delta n \tag{3.18}$$

Finally, from boundary condition (2.1) we readily obtain

$$\int_{t_1}^{t_2} dt\delta\sigma S = \int_{t_1}^{t_2} dt\int_S (p_- - p_+)\,\delta n\,dS$$

where S is the drop surface area.

Adding equalities (3.17) and (3.18) and subtracting from the sum the last equality, we obtain

$$\int_{t_1}^{t_2} dt\delta L = 0 \tag{3.19}$$

$$L(q_i, q_i^{\cdot}) = T_+ + T_- - \sigma S - \rho_-\omega\int_V (\psi-\psi^*)\,d\tau$$

Extremum of the variational equation (3.19) is the solution of the system of Lagrange differential equations.

The considered derivation of Lagrange equations can be extended to more general cases of fluid motion in volume V bounded by a deformable surface S.

1°. Potential cyclic motion in an arbitrary M-connected region V. Region V is transformed into a simply connected one by the addition of $M-1$ partitions Π_k $(k = 1, 2, \ldots\ M-1)$. Then

$$\int_{t_1}^{t_2} dt\delta L = \int_{t_1}^{t_2} dt\int_S p\delta n\,dS,\quad L = T - \sum_{k=1}^{M-1}\Gamma_k Q_k^{\cdot} \tag{3.20}$$

$$Q_k^{\cdot} = \rho\int_{\Pi_k} (\mathbf{v}-\mathbf{v}^*)\,\mathbf{n}\,dS$$

where T is the kinetic energy of fluid in volume V, Γ_k are circulations along mutually nontransferable contours, and Q_k are the flow rates through partitions.

2°. Plane-parallel cyclic motion with constant vortex ω and a similar axisymmetric flow in an M-connected region. The Lagrange function is of the form

$$L = T - \sum_{k=1}^{M-1}\Gamma_k Q_k^{\cdot} - \rho\omega\int_S (\psi-\psi^*)\,d\tau \tag{3.21}$$

In the first case the Lagrange equations can be derived with the use of the identity

$$\frac{d}{dt}\int_{\Pi_k} (\boldsymbol{\xi}-\boldsymbol{\xi}^*)\,\mathbf{n}\,dS = \delta\int_{\Pi_k} (\mathbf{v}-\mathbf{v}^*)\,\mathbf{n}\,dS \tag{3.22}$$

Note that owing to conditions (3.14) the integrals over partitions are independent of the selection of these, hence the law of motion of the partition can be arbitrarily specified. Let us assume that partitions Π_k move together with fluid. To prove the above identity we use the formula for the derivative of the solenoidal vector stream through a fluid surface with respect to time [13], and the similar formula for variations

$$\frac{d}{dt}\int\limits_{\Pi_k}(\xi-\xi^*)\,\mathbf{n}\,dS=\int\limits_{\Pi_k}\left[\frac{\partial}{\partial t}(\xi-\xi^*)+\operatorname{rot}((\xi-\xi^*)\times\mathbf{v})\right]\mathbf{n}\,dS \tag{3.23}$$

$$\delta\int\limits_{\Pi_k}(\mathbf{v}-\mathbf{n}^*)\,\mathbf{n}\,dS=\int\limits_{\Pi_k}[\delta(\mathbf{v}-\mathbf{v}^*)+\operatorname{rot}((\mathbf{v}-\mathbf{v}^*)\times\xi)]\,\mathbf{n}\,dS$$

Using Eqs. (3.3), (3.4) and (3.10) we obtain from this

$$\frac{d}{dt}\int\limits_{\Pi_k}(\xi-\xi^*)\,\mathbf{n}\,dS-\delta\int\limits_{\Pi_k}(\mathbf{v}-\mathbf{v}^*)\,\mathbf{n}\,dS=\int\limits_{\Pi_k}\operatorname{rot}[(\xi-\xi^*)\times(\mathbf{v}-\mathbf{v}^*)]\,\mathbf{n}\,dS=$$

$$\oint\limits_{L_k}(\xi-\xi^*)\times(\mathbf{v}-\mathbf{v}^*)\,d\mathbf{l}$$

where L_k is the contour bounding partition Π_k. Contour L_k lies on surface S. Vectors $\xi-\xi^*$, $\mathbf{v}-\mathbf{v}^*$ and $d\mathbf{l}$ lie in one plane (by virtue of (3.14) they are perpendicular to a single vector $\mathbf{n}$), hence the last integral is zero, which proves the identity (3.22). After some obvious transformations and integration of identity (3.22) with respect to time, we obtain

$$\int\limits_V\rho\mathbf{v}\xi\,d\tau\,|_{t_2}=\int\limits_V\operatorname{div}\rho\Phi\xi\,d\tau\,|_{t_2}=\sum_{k=1}^{M-1}\Gamma_k\int\limits_{\Pi_k}\xi\mathbf{n}\,dS\,|_{t_2}=\int\limits_{t_1}^{t_2}dt\,\delta\sum_{k=1}^{M-1}\Gamma_kQ_k^{\cdot}$$

From this and the variational equation (3.7) we obtain (3.20).

To derive the Lagrange equations (3.21) it is sufficient to use the equality

$$\int\limits_V\rho\mathbf{v}\xi\,d\tau\,|_{t_1}^{t_2}=\int\limits_{t_1}^{t_2}dt\,\delta\left(\sum_{k=1}^{M-1}\Gamma_kQ_k^{\cdot}+\rho\omega\int\limits_V(\psi-\psi^*)\,d\tau\right)$$

which is obtained similarly to (3.16).

Note that Lagrange equations for the plane case are formally derived by substituting unity for y in equations of the axisymmetric case.

4. Variational equation of steady motion. Condition of stability. In the case of motion of a drop in an unbounded fluid the Lagrange function is independent of coordinate x_0, which determines its translation along the axis of symmetry. Since the coordinate x_0 is cyclic, it is possible to introduce the Routh function L°.

$$L^\circ=L-u\frac{\partial L}{\partial u} \tag{4.1}$$

We eliminate velocity $u=x_0^{\cdot}$ by using the law of conservation of momentum

$$\partial L/\partial u=P \tag{4.2}$$

The Routh function L° is the Lagrange function of the reduced system with position coordinates $q_1,\ldots,y_N$, which define the shape of the drop [14]. The term L°, which is independent of generalized velocities $q_i^{\cdot}$, taken with the opposite sign is the potential energy U of the reduced system. To calculate U it is sufficient to set $q_i^{\cdot}=0$ in formulas (3.19), (4.1) and (4.2). Taking into account that

$$\psi=\omega\psi_\omega,\quad\psi^*=0 \tag{4.3}$$

$$T_+=\tfrac{1}{2}Mu^2,\qquad T_-=\tfrac{1}{2}(\rho_-Vu^2+I\omega^2)$$

$$I = \rho_- \int_V \left(\operatorname{rot} \mathbf{k} \frac{\psi_\omega}{y}\right)^2 d\tau = \rho_- \int_V \psi_\omega \, d\tau$$

where M is the apparent mass of the drop and ψ_ω is the streamfunction of the steady vortex motion inside the drop, being the solution of the following boundary value problem [11]:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{y}\frac{\partial}{\partial y}\right)\psi_\omega = -y^2, \quad \psi_\omega|_S = 0 \tag{4.4}$$

hence it is possible to obtain for U the following formula:

$$U = \frac{P^2}{2(M + \rho_- V)} + \frac{1}{2} I\omega^2 + \sigma S \tag{4.5}$$

The condition of equilibrium of the reduced system, viz. $\delta U = 0$, corresponds to the condition of steady motion of the drop. The condition of stability of that motion is $\delta^2 U > 0$. Effecting related variations at constant volume of the drop, we obtain the condition of stability of steady motion of the drop

$$-{}^1/_2 u^2 \,\delta M + {}^1/_2\, \omega^2 \delta I + \sigma \delta S = 0 \tag{4.6}$$

$$-\frac{1}{2} u^2 \delta^2 M + \frac{u^2 (\delta M)^2}{(M + \rho_- V)} + \frac{1}{2} \omega^2 \delta^2 I + \sigma \delta^2 S > 0$$

The variational equation for the steady motion of the drop may be obtained directly without the use of Lagrange equations.

For the variation of apparent mass [12] and of surface area we have the following formulas:

$$\delta \frac{1}{2} u^2 (M + \rho_+ V) = \frac{\rho_+}{2} \int_S v_+^2 \delta n \, dS \tag{4.7}$$

$$\delta S = \int_S 2H \delta n \, dS$$

To obtain similar formula for δI we use the identity

$$\int_V (\mathbf{A} \operatorname{rot} \operatorname{rot} \mathbf{B} - \mathbf{B} \operatorname{rot} \operatorname{rot} \mathbf{A})\, d\tau = \int_S (\mathbf{B} \times \operatorname{rot} \mathbf{A} - \mathbf{A} \times \operatorname{rot} \mathbf{B})\, \mathbf{n}\, dS$$

Setting in it

$$\mathbf{A} = \mathbf{k} \frac{\delta\psi_\omega}{y}, \quad \mathbf{B} = \mathbf{k} \frac{\psi_\omega}{y}$$

and taking into account that ψ_ω is the streamfunction for the motion of fluid with unit vorticity

$$\operatorname{rot} \operatorname{rot} \mathbf{A} = 0, \quad \operatorname{rot} \operatorname{rot} \mathbf{B} = \mathbf{k} y, \quad \psi_\omega|_S = 0$$

we obtain

$$\int_V \delta\psi_\omega \, d\tau = -\int_S \frac{\delta\psi_\omega}{y} \mathbf{k} \times \operatorname{rot} \mathbf{k} \frac{\psi_\omega}{y} \mathbf{n}\, dS = -\int_S \frac{\delta\psi_\omega}{y^2} \frac{\partial\psi_\omega}{\partial n} dS \tag{4.8}$$

The Lagrangian variation ψ_ω on surface S is zero, since ψ_ω is identically zero on S, hence from (3.2) follows that

$$-\delta\psi_\omega = \frac{\partial\psi_\omega}{\partial n} \delta n$$

Substituting Euler's variation $\delta\psi_\omega$ into (4.8), we obtain the required formula

$$\frac{1}{2} \omega^2 \delta I = \frac{\rho_-}{2} \int_S v_-^2 \delta n \, dS \tag{4.9}$$

From the Bernoulli integral and the boundary condition (2.2) follows that

$$\int_S \left(\frac{\rho_+}{2} v_+^2 - \frac{\rho_-}{2} v_-^2 - 2H\sigma\right) \delta n \, dS = \int_S \text{const} \, \delta n \, dS \tag{4.10}$$

The variational equation (4.6) is readily obtained from (4.8) — (4.10) with an accuracy to within $c\delta V$.

Since for the determination of the drop shape by the variational equation (4.6) $\delta V = 0$ is assumed, hence constant c is immaterial.

The ellipsoidal drop. The complete investigation of the drop dynamics necessitates the introduction of an infinite number of generalized coordinates. However the variational equations (3.13) and (4.6) make it possible to obtain approximate solutions of this problem. For this it is sufficient to consider some class of surfaces S with a finite number of degrees of freedom.

Let the drop surface have the form of an ellipsoid of revolution

$$\frac{(x-x_0)^2}{l_x^2} + \frac{y^2}{l_y^2} - 1 = 0, \quad l_x l_y^2 = l^3 \tag{5.1}$$

The position of surface S of ellipsoid (5.1) may be specified by two generalized coordinates x_0 of the center and $\chi = l_y / l_x$ which is the ratio of the ellipsoid axes.

To calculate the Lagrange functioh (3.19) it is necessary to determine: (1) the kinetic energy of the external potential motion, (2) the kinetic energy of fluid motion with constant vorticity inside the drop, and (3) the streamfunction for the motion of the medium attached to the body and the streamfunction for the flow of fluid inside the drop.

Solution of the first problem is given in [8]. We pass to the solution of the second and third problems.

Motion of the medium attached to the body may be specified by the superposition of two transformations

$$x = x_0 + x'; \quad x' = l\chi^{-2/3}x^*$$
$$y = y'; \quad y' = l\chi^{1/3}y^*$$

The field of velocity v^* determined by (3.9) is

$$v_x^* = u - \frac{2}{3}\frac{\chi^\cdot}{\chi}x', \quad u = x_0^\cdot \tag{5.2}$$
$$v_y^* = \frac{1}{3}\frac{\chi^\cdot}{\chi}y'$$

It is not difficult to ascertain that the velocity field (5.2) is solenoidal and also potential. The streamfunction ψ^* for that field is

$$\psi^* = \frac{1}{2}uy^2 - \frac{1}{3}\frac{\chi^\cdot}{\chi}x'y^2 \tag{5.3}$$

Function ψ may be represented by the sum

$$\psi = \psi^* + \omega\psi_\omega \tag{5.4}$$

where ψ_ω is obtained by solving the boundary value problem (4.4). For the ellipsoid (5.1) the solution of that problem is of the form

$$\psi_\omega = \frac{l^2}{2}\frac{\chi^{2/3}}{\chi^2+4}y^2\left(1 - \frac{x^2}{l_x^2} - \frac{y^2}{l_y^2}\right) \tag{5.5}$$

which for $\chi = 1$ yields the known streamfunction for the spherical Hill vortex [11].

For the determination of kinetic energy of the drop it is convenient to use the equalities

$$\mathbf{v} = \mathbf{v}_* + \mathbf{v}_\omega, \quad \mathbf{v}^2 = \mathbf{v}_*^2 + \mathbf{v}_\omega^2 + 2\,\mathrm{div}\,(\Phi_* \mathbf{v}_\omega)$$

where $\mathbf{v}_\omega$ is the velocity field of the fluid whose streamfunction is $\omega\psi_\omega$ and Φ_* is the potential of the velocity field $\mathbf{v}_*$.

Integrating the last equality over the drop volume and taking into consideration that vector $\mathbf{v}_*$ is perpendicular to the normal to surface S, for the kinetic energy of the drop we obtain

$$T_- = \frac{\rho_-}{2}\int\limits_V (\mathbf{v}_*^2 + \mathbf{v}_\omega^2)\, d\tau$$

Finally, using the identity

$$\mathbf{v}_\omega^2 = \omega\left(\omega\psi_\omega + \mathrm{div}\left(\frac{\mathbf{k}}{y}\,\psi_\omega \times \mathbf{v}_\omega\right)\right)$$

we obtain the definitive formula for the kinetic energy of the drop

$$T_- = T_-^* + \frac{1}{2}\omega^2 I, \quad T_-^* = \frac{\rho_-}{2}\int\limits_V \mathbf{v}_*^2 d\tau, \quad I = \rho_- \int\limits_V \psi_\omega d\tau \tag{5.6}$$

where T_-^* is the kinetic energy of fluid inside the drop in the absence of vorticity; it is equal the sum of the kinetic energy of the drop moving as a solid at velocity u and of the kinetic energy due to the deformation of the drop boundary. Substituting (5.6) into (3.19), we obtain the Lagrange function

$$L = T_+ + T^*_- - {}^1\!/_2\omega^2 I - \sigma S, \quad T_+ = {}^1\!/_2 M u^2 \tag{5.7}$$

$$M = \frac{4\pi}{3}\rho_+ l^3 m(\chi), \quad I = \frac{4\pi}{3}\rho_- l^7 i(\chi), \quad S = \frac{2\pi}{3} l^7 s(\chi)$$

$$m = \frac{1-a}{a - 1/\chi^2}, \quad i = \frac{2\chi^{4/3}}{35(\chi^2+4)}, \quad s = 3(\chi^{2/3} + \chi^{-4/3} b)$$

$$a = \int\limits_0^1 \frac{dx}{1 + (\chi^2 - 1)x^2}, \quad b = \int\limits_0^1 \frac{dx}{1 + (\chi^{-2} - 1)x^2}$$

where functions $a(\chi)$ and $b(\chi)$ are expressed in the intervals $0 < \chi < 1$ or $1 < \chi < \infty$ in terms of inverse trigonometric or hyperbolic functions. Substituting functions (5.7) into (4.6), we obtain for the stable steady motion of the ellipsoidal drop the following conditions:

$$-m' + \Omega\lambda i' + (1/W)\,s' = 0 \tag{5.8}$$

$$-m'' + \frac{2m'^2}{m+\lambda} + \Omega\lambda i'' + \left(\frac{1}{W}\right)s'' > 0$$

$$\Omega = \frac{\omega^2 l^4}{u^2}, \quad \lambda = \frac{\rho_-}{\rho_+}, \quad W = \frac{u^2\rho_+ l}{\sigma}$$

where primes denote derivatives with respect to χ.

System (5.8) shows that the stable "equilibrium" of the drop depends on three dimensionless quantities, and not on the single Weber number, as is usually assumed [4].

It should be noted that inequality (5.8) is the necessary condition of stability but is not sufficient, since stability with respect to perturbations of a special form and with the drop retaining its ellipsoidal shape is considered here. Reversal of the inequality sign yields the condition of the drop instability.

In the system of coordinates $1 / W$, $\Omega\lambda$ (Fig. 1) the first of Eqs. (5.8) defines a one-parameter set of straight lines which determine all states of equilibrium of the drop. Each straight line is tangent to the envelope of the set which is represented in Fig. 1 by the heavy solid line

$$f = 0, \quad \partial f / \partial \chi = 0 \qquad (5.9)$$
$$f(\chi, \Omega\lambda, 1 / W) = -m' + \Omega\lambda i' + (1 / W) s'$$

To every point of the envelope corresponds a single completely determinate value of parameter χ such that the tangent at any of its points represents the line of the drop equilibrium state for the same value of parameter χ. In Fig. 1 values of $\chi = 0.2, 0.4, 0.6, \ldots, 3.6$ are indicated by short cross lines.

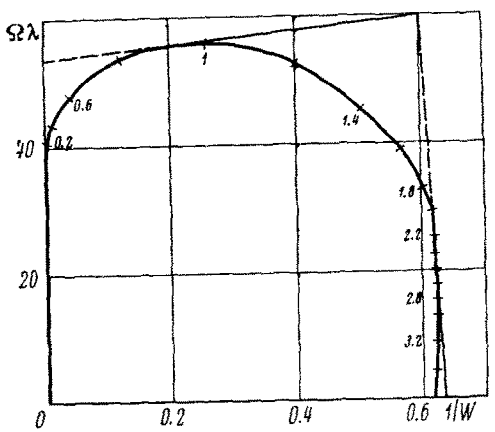

Fig. 1

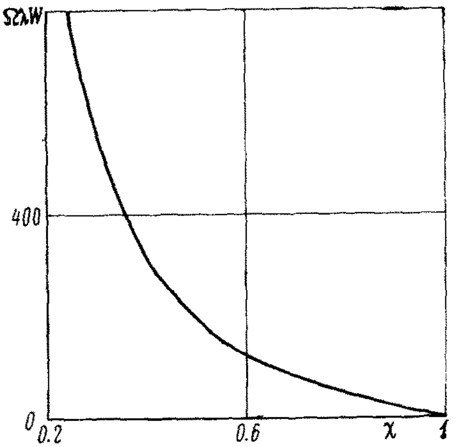

Fig. 2

The dependence of the angle of slope of a tangent to the envelope on χ is defined by function $-s' / i'$ and is shown in Fig. 2. The monotonic decrease of that function indicates that the envelope is convex.

The envelope separates the coordinate plane in Fig. 1 into two regions. In the first of these bounded by the envelope and the abscissa axis no equilibrium states are possible. If the point defined by coordinates $1 / W$, $\Omega\lambda$ lies in the second region, the extent of drop deformation in the equilibrium state can be determined by drawing from the point a tangent to the envelope, and the tangency point determines the value of parameter χ. To each pair of numbers $1 / W$, $\Omega\lambda$ correspond two equilibrium states with different axes ratio χ. For example, when $1 / W = 0.6$ and $\Omega\lambda = 60$, the drop has two equilibrium states with $\chi \approx 0.9$ and $\chi \approx 2.3$.

Stability of the equilibrium shape of the drop is determined by inequality (5.8) and depends on three dimensionless quantities, viz. $1 / W$, $\Omega\lambda$ and λ. At each of the straight lines of equilibrium states with a constant χ there exists a point which separates stable states from unstable ones. The position of that point depends on λ.

6. The case of $\lambda \gg 1$. The motion of a liquid drop in gas belongs to this category. A fairly simple geometrical interpretation of the stable states of the drop can be given in this case.

At the limit $\lambda \to \infty$ with $\Omega\lambda = \mathrm{const}$ we obtain from system (5.8)

$$f(\chi, \Omega\lambda, 1 / W) = 0, \quad \partial f / \partial \chi > 0 \tag{6.1}$$

At each straight line of the sets of equilibrium states function $\partial f / \partial \chi$ changes its sign at the point of tangency with the envelope and, consequently, in conformity with (6.1) a change of the stability state takes place. The state of the drop is stable when the vector drawn from the tangency point to point $1 / W$, $\Omega\lambda$ is pointing in the direction of increasing parameter χ, and unstable when that vector is pointed in the opposite direction.

In Fig. 1 stable states are shown by the solid straight line and the unstable ones by the hatched line. For example, when $1 / W = 0.6$ and $\Omega\lambda = 60$, $\chi \approx 0.9$ and $\chi \approx 2.3$ correspond to stable and unstable states, respectively.

It is seen from Fig. 1 that for $\Omega\lambda \gtrsim 56$ the extent of drop deformation is defined by $\chi \leqslant 1$, i.e. the drop is elongated in the direction of the flow velocity. In the opposite case, when $\Omega\lambda \lesssim 56$, $\chi \geqslant 1$ and the drop is compressed in the direction of the flow velocity.

When condition (1.4) is satisfied, the order of magnitude of parameter $\Omega\lambda$ can be estimated by (1.5) as

$$\Omega\lambda \sim \frac{m}{i} \frac{\rho_+ v_+^2}{\rho_- v_-^2} R_+$$

The ratio m / i varies from 40 when $\chi \approx 0.2$ to 120 when $\chi \approx 3.5$. When $\chi = 1$, $m / i \approx 44$. For the estimate we assume $m / i \approx 50$, then

$$\Omega\lambda \sim 50 \frac{\rho_+ v_+^2}{\rho_- v_-^2} R_+ \tag{6.2}$$

In the case of a water drop in air $\Omega\lambda \sim 15 R_+$, hence at high Reynolds numbers water drops are elongated in the direction of the air flow.

Three particular cases of motion of drops merit consideration. They are as follows:

1) the kinetic energy of the fluid outside the drop is considerably lower than the sum of the drop kinetic energy and of the potential energy of surface tension $T_+ \ll T_- + \sigma S$;

2) the kinetic energy of the drop is considerably lower than the sum of kinetic energy outside the drop and the potential energy of surface tension $T_- \ll T_+ + \sigma S$;

3) the potential energy of the drop surface tension is considerably lower than the sum of kinetic energies of fluid inside and outside the drop $\sigma S \ll T_- + T_+$.

The state of equilibrium in these three cases is completely determined by a single dimensionless parameter, whose dependence on χ is readily obtained from the equilibrium equation (6.1) by neglecting the first, second and third terms, respectively

$$1)\ \Omega\lambda W = -\frac{s'}{i'}, \quad 2)\ W = \frac{s'}{m'}, \quad 3)\ \Omega\lambda = \frac{m'}{i'} \tag{6.3}$$

The curve of the first function in (6.3) plotted in Fig. 2 shows that the equilibrium shape of all drops is an elongated one ($\chi \leqslant 1$).

In this case the condition of stability is obtained by substituting the first of formulas (6.3) into (6.1) and neglecting the term m''

$$\frac{d}{d\chi} \frac{s'}{i'} > 0$$

Since function s' / i' monotonically increases (Fig. 2), all states of equilibrium of the drop are stable.

These results may be obtained by geometric means with the use of Fig. 1, by taking into consideration that at the limit when one of the parameters $\Omega\lambda$ or $1 / W$ tends to infinity, the lines of equilibrium state are tangent to the envelope at points $\chi \leqslant 1$. All these states are stable.

The second limit case of (6.3) of absence of flow inside the drop was investigated in [6]. Function $W(\chi)$ monotonically increases up to $\chi \lessapprox 3.7$, reaches its maximum $W \approx 1.63$, and then monotonically decreases. A similar, but apparently less exact form of function $W(\chi)$ was derived in [5] by another method.

Note 6.1. The above analysis implies that the equilibrium state of the drop is stable even when $\chi \to 0$, which is at variance with the physical meaning. This is explained by that the state of the drop is stable with respect to perturbations for which it retains its ellipsoidal shape, but may lose its stability when subjected to some other perturbation. Fairly elongated drops will be evidently unstable with respect to perturbations of the form of radial expansions and compressions in the equatorial plane of the drop, as is the case of capillary jets which in the Rayleigh theory are unstable [12].

It is thus necessary to supplement the condition of stability by the condition that $\chi > \chi_*$, where χ_* is a certain critical value of the drop deformation.

Note 6.2. The second case of (6.3) obtains when a quiescent drop is hit by a stream of gas. The characteristic time of establishment of a vortex flow inside the drop is $t \sim l^2 / \nu_-$, in the course of which it covers the distance $ut \sim lR_+\nu_+ / \nu_-$. For a water drop in the air that distance exceeds the drop dimension a thousand times, and may substantially exceed the distance passed by the drop during the observation time. The vortex inside the drop cannot develop and may, therefore, be neglected.

Note 6.3. Function $W(\chi)$ was independently determined for $\Omega\lambda = 0$ in [15, 16] in the solution of the problem of motion of an ellipsoidal bubble, using the same methods as in [5, 6], respectively. The problem of stability was not considered in [4 – 6]. The statement about the instability of the fixed shape of a drop along the decreasing section of function $W(\chi)$ is not obvious and valid at the limit $\lambda \to \infty$. It was shown in [16] that all equilibrium shapes are stable.

It will be seen from Fig. 1 that at the limit $\Omega\lambda \doteq 0$ stable states of the drop occur when $W < 1.63$ and $\chi < 3.7$. The same result can be obtained from the stability condition (6.1) by substituting into the inequality the second of formulas (6.3)

$$\frac{d}{d\chi} W(\chi) > 0$$

This the increasing section of function $W(\chi)$ corresponds to stable states of the drop.

Analysis of the third case of (6.3) which corresponds to the limit $1 / W \to 0$ shows that all states of the drop are unstable. This is the case of fairly large drops in which the stabilizing effect of surface tension on the drop stability is negligibly small. It is obviously natural that such drops cannot be in a stable state.

The author thanks L. I. Sedov for discussing the results of this work, and also P. A. Petrosian and O. V. Voinov for useful remarks.

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Translated by J. J. D.